

A Note on CFT Correlators in Three Dimensions

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Abstract

In this note we present a simple method of constructing general conformally invariant three point functions of operators of various spins in three dimensions. Upon further imposing current conservation conditions, we find new parity violating structures for the three point functions involving either the stress-energy tensor, spin one currents, or higher spin currents. We find that all parity preserving structures for conformally invariant three point functions of higher spin conserved currents can be realized by free fields, whereas there is at most one parity violating structure for three point functions for each set of spins, which is not realized by free fields.

1 Introduction

The holographic duality between quantum theories of gravity in AdS_4 and three dimensional CFTs [1] is particularly rich and intriguing for several reasons. Firstly, it provides in principle a complete description of quantum gravity in four dimensions. Such a description could be used to address the existence of UV completions of pure gravity or gravity coupled to a few matter fields in four dimensions, as well as to provide a computable framework for gravity interactions at (super-)Planckian scale. Secondly, three dimensional CFTs are extraordinarily rich. In particular, there is a large family of 3d CFTs with the Lagrangian description in terms of Chern-Simons-matter (CSM) theories [2]. Generally, these theories can be studied perturbatively. In some cases, exact nonperturbative results are also possible (such as theories with vector-like matter in the large N limit, or certain BPS observables in supersymmetric theories that can be computed via localization method). Thirdly, there are conjectures of *exact* dualities between nonsupersymmetric vector-type 3d CFTs and higher spin gravity theories in AdS_4 [3, 4]. Strong evidences in support of these dualities have been found recently [5, 6], and a computable framework has been formulated which allows one to compare the boundary and bulk order by order in the $1/N$ expansion.

Thus far, the AdS/CFT correspondence has been mostly useful only when the boundary theory has a $1/N$ expansion, corresponding to the perturbative expansion in the bulk gravity coupling. A “typical” large N CFT has primary operators of various spins and dimensions that do not scale with N . These operators are dual to various higher spin fields in the bulk. In this note, we will explore constraints on correlation functions of higher spin operators in a 3d CFT, from conformal symmetry alone, extending earlier work of Osborn and Petkou [8]. Although we do not expect interacting 3d CFTs to have exact higher spin symmetries, sometimes there can be enhanced higher spin symmetries in the infinite N limit, which implies that certain correlators obey Ward identities for conserved currents at the leading nontrivial order in the $1/N$ expansion. We will consider both conserved and non-conserved higher spin currents, as well as spin one and spin two currents. We will focus on the three point functions. In particular, in the case of spin 1 and spin 2 conserved currents (denote by j_μ and $T_{\mu\nu}$), we find new, parity violating tensor structures for the three point functions $\langle Tjj \rangle$ and $\langle TTT \rangle$, which were not given in the previous work of Osborn and Petkou [8] because parity has been implicitly assumed there. It will be demonstrated in [10] that these parity odd structures are indeed present in parity violating Chern-Simons-matter theories. Parity violating structures are found for higher spin correlators as well.

In the next section, we describe the general conformally invariant three point functions without imposing current conservation. We then present a conjectural classifica-

tion of all conformally invariant three point functions of conserved higher spin currents. Our main results are summarized in section 3.1. Various examples and tests are presented in subsequent sections. We conclude with some comments on the holographic duals.

Note: Upon completion of the bulk of this work, we became aware of [11] which also observed the parity violating $\langle TTT \rangle$ structure.

2 Correlators of higher spin operators

2.1 Constraints from conformal symmetry

Denote by $J_{\mu_1 \dots \mu_s}(x)$ a primary operator of spin s . In particular, $J_{\mu_1 \dots \mu_s}$ is symmetric and traceless in (μ_1, \dots, μ_s) . It is convenient to work with three dimensional spinor indices, and write the operator as $J_{\alpha_1 \dots \alpha_{2s}} = J_{\mu_1 \dots \mu_s} \sigma_{\alpha_1 \alpha_2}^{\mu_1} \dots \sigma_{\alpha_{2s-1} \alpha_{2s}}^{\mu_s}$, which is completely symmetric in $(\alpha_1, \dots, \alpha_{2s})$. In this section, we will not assume J to be a conserved current, and only use constraints from conformal symmetry. It will turn out that the constraints from conformal symmetry is easy to implement in our approach, while the constraints from current conservation is more complicated. The consequence of imposing current conservation will be explored in the next section.

It is convenient to write the higher spin current in terms of a generating function, by contracting with polarization spinor λ^α ,

$$J_s(x, \lambda) = J_{\alpha_1 \dots \alpha_{2s}}(x) \lambda^{\alpha_1} \dots \lambda^{\alpha_{2s}} \quad (2.1)$$

This is equivalent to contracting $J_{\mu_1 \dots \mu_s}$ with a null polarization vector ε , related to λ by $\varepsilon_{\alpha\beta} \equiv \varepsilon_\mu \sigma_{\alpha\beta}^\mu = \lambda_\alpha \lambda_\beta$. Consider the n -point function of currents of various spins,

$$F(\{x_i, \lambda_i, s_i\}) = \left\langle \prod_{i=1}^n J_{s_i}(x_i, \lambda_i) \right\rangle \quad (2.2)$$

For now we will not assume that the currents are conserved, but will assume that each J_s has scaling dimension $\Delta = s + 1$. The latter saturates the unitarity bound of the conformal algebra, and for a unitary CFT in fact implies the conservation of the current. The analysis of this section is not affected much by this assumption. If $J_{s_i}(x_i)$ has dimension $\Delta_i = s_i + 1 + \delta_i$ for some nonzero δ_i , then the form of the correlation function (as constrained by conformal symmetry) would simply be multiplied by that of scalar primaries of dimension δ_i inserted at x_i .

The conformal group is generated by the Poincaré group together with dilatation D and special conformal generators K_μ . K_μ can be realized as $RP_\mu R$, where R is the inversion $x^\mu \mapsto x^\mu/x^2$. Hence we look for Poincaré invariant structures with fixed weight under D and invariant under R up to a sign. Invariance under D implies that $F(\{x_i, \lambda_i\})$ is homogeneous under the rescaling

$$x_i \mapsto tx_i, \quad \lambda_i \mapsto t^{\frac{1}{2}}\lambda_i, \quad (2.3)$$

of degree $-n$. Under the inversion R , we have

$$x^\mu \mapsto \check{x}^\mu \equiv \frac{x^\mu}{x^2}, \quad \lambda \mapsto \check{\mathbf{x}}\lambda \equiv \check{\lambda}, \quad (2.4)$$

where $\mathbf{x} \equiv \vec{x} \cdot \vec{\sigma}$, $\check{\mathbf{x}} = \mathbf{x}/x^2$. The action by R on λ is defined such that $\varepsilon^\mu = \lambda \sigma^\mu \lambda$ transforms as

$$\varepsilon^\mu \mapsto \check{\varepsilon}^\mu = \frac{\partial \check{x}^\mu}{\partial x^\nu} \varepsilon^\nu = \frac{\varepsilon^\mu}{x^2} - \frac{2\varepsilon \cdot x x^\mu}{x^4}, \quad (2.5)$$

and so the spin- s current transforms as $J_s(x, \lambda) \rightarrow \check{J}_s(\check{x}, \check{\lambda})$, with

$$\check{J}_s(\check{x}, \check{\lambda}) = x^2 J_s(x, \lambda) \quad (2.6)$$

Invariance of the correlation function under R then implies

$$F(\{\check{x}_i, \check{\lambda}_i, s_i\}) = \left(\prod_{i=1}^n x_i^2 \right) F(\{x_i, \lambda_i, s_i\}) \quad (2.7)$$

For given spins s_i , a priori, F depends on $5n$ variables (x_i, λ_i) . The conformal group in three dimensions, $SO(3, 2)$, has 10 generators. We thus expect to express F in terms of $5n - 10$ variables. From now on, we will focus on the $n = 3$ case, where F depends on 5 independent variables, which should be expressed as Poincaré invariant combinations of $\{x_i, \lambda_i\}$.

Note that under R , \mathbf{x}_{ij} and x_{ij}^2 transform as

$$\begin{aligned} \mathbf{x}_{ij} &\rightarrow \frac{\mathbf{x}_i}{x_i^2} - \frac{\mathbf{x}_j}{x_j^2} = -\check{\mathbf{x}}_i \mathbf{x}_{ij} \check{\mathbf{x}}_j, \\ x_{ij}^2 &\rightarrow x_{ij}'^2 = \frac{x_{ij}^2}{x_i^2 x_j^2}, \\ \check{\mathbf{x}}_{ij} &\rightarrow -\mathbf{x}_i \check{\mathbf{x}}_{ij} \mathbf{x}_j. \end{aligned} \quad (2.8)$$

So expressions of the form

$$\lambda_{i_1} \check{\mathbf{x}}_{i_1 i_2} \mathbf{x}_{i_2 i_3} \cdots \check{\mathbf{x}}_{i_{n-1} i_n} \lambda_{i_n} \quad (2.9)$$

are invariant under the action of R (note that $(\mathbf{x}\lambda)^T = -\lambda^T \mathbf{x}$, and we have omitted the transpose in writing $\lambda \cdots \lambda$ with the understanding that the spinor indices are contracted from upper left to lower right). Therefore we see that the following six Poincaré invariant objects

$$\begin{aligned} P_3 &= \lambda_1 \check{\mathbf{x}}_{12} \lambda_2, \quad P_1 = \lambda_2 \check{\mathbf{x}}_{23} \lambda_3, \quad P_2 = \lambda_3 \check{\mathbf{x}}_{31} \lambda_1, \\ Q_1 &= \lambda_1 \check{\mathbf{x}}_{12} \mathbf{x}_{23} \check{\mathbf{x}}_{31} \lambda_1, \quad Q_2 = \lambda_2 \check{\mathbf{x}}_{23} \mathbf{x}_{31} \check{\mathbf{x}}_{12} \lambda_2, \quad Q_3 = \lambda_3 \check{\mathbf{x}}_{31} \mathbf{x}_{12} \check{\mathbf{x}}_{23} \lambda_3. \end{aligned} \quad (2.10)$$

are invariant under inversion, hence conformally invariant. We can alternatively write the Q_i 's as

$$Q_1 = -\lambda_1 (\check{\mathbf{x}}_{12} + \check{\mathbf{x}}_{31}) \lambda_1, \quad Q_2 = -\lambda_2 (\check{\mathbf{x}}_{23} + \check{\mathbf{x}}_{12}) \lambda_2, \quad Q_3 = -\lambda_3 (\check{\mathbf{x}}_{31} + \check{\mathbf{x}}_{23}) \lambda_3. \quad (2.11)$$

In vector notations, these objects correspond to the following conformally invariant tensor structures

$$(P_3)^2 = -2\varepsilon_1^\mu \varepsilon_2^\nu \left[\frac{\delta^{\mu\nu}}{x_{12}^2} - 2 \frac{x_{12}^\mu x_{12}^\nu}{x_{12}^4} \right], \quad Q_1 = 2\varepsilon_1^\mu \left[\frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{12}^\mu}{x_{12}^2} \right], \quad (2.12)$$

and similarly for the remaining structures related to these by cyclic permutations. In particular $(P_3)^2$ is the structure appearing in the 2-point function of higher-spin currents, $\langle J_s(x_1, \lambda_1) J_s(x_2, \lambda_2) \rangle = c |x_{12}|^{-2} (P_3)^{2s}$, for some normalization constant c .

Any polynomial of $\{P_i, Q_i\}$ that is homogeneous of degree $2s_i$ with respect to λ_i , when multiplied by $|x_{12}|^{-1} |x_{23}|^{-1} |x_{31}|^{-1}$, gives a tensor structure for $\langle J_{s_1} J_{s_2} J_{s_3} \rangle$ that is allowed by conformal symmetry. Note that these P_i, Q_i 's are precisely the building blocks of the generating function of 3-point functions of higher spin currents in the free massless scalar and free massless fermion theory, which are respectively [6]

$$\begin{aligned} \mathcal{F}_{\langle JJJ \rangle}^{\text{bose}} &= \frac{1}{|x_{12}| |x_{23}| |x_{31}|} \cosh \left[\frac{1}{2} (Q_1 + Q_2 + Q_3) \right] \cosh P_1 \cosh P_2 \cosh P_3 \\ \mathcal{F}_{\langle JJJ \rangle}^{\text{fermi}} &= \frac{1}{|x_{12}| |x_{23}| |x_{31}|} \sinh \left[\frac{1}{2} (Q_1 + Q_2 + Q_3) \right] \sinh P_1 \sinh P_2 \sinh P_3. \end{aligned} \quad (2.13)$$

The LHS are generating functions of $\langle J_{s_1} J_{s_2} J_{s_3} \rangle$ summed over the spins. The free scalar expression is valid for $s_i \geq 0$ (the spin 0 “current” is just the dimension 1 operator $\bar{\phi}\phi$), while the free fermion answer is valid for $s_i \geq 1$ (see section 3.3 below for the form of the generating function involving the $\Delta = 2$ scalar operator $\bar{\psi}\psi$).

As remarked earlier, we expect that the 3-point functions should depend on $15 - 10 = 5$ independent conformally invariant objects. Indeed, it can be seen that the six structures (P_i, Q_i) satisfy the following non-linear relation

$$P_1^2 Q_1 + P_2^2 Q_2 + P_3^2 Q_3 - 2P_1 P_2 P_3 - Q_1 Q_2 Q_3 = 0, \quad (2.14)$$

thus leaving 5 independent degrees of freedom.

In theories with a parity symmetry, $J_{\mu_1 \dots \mu_s}$ acquires a sign $(-)^s$ under parity. We will assign λ to transform as $\lambda \rightarrow i\lambda$ under parity, so that $J(x, \lambda)$ would be parity invariant. The P_i, Q_i structures defined above are even under the action of parity $x_k^\mu \rightarrow -x_k^\mu$, $\lambda_k \rightarrow i\lambda_k$, for $k = 1, 2, 3$. There are further parity odd conformal invariant structures that one can construct using the antisymmetric tensor $\epsilon_{\mu\nu\rho}$. They are¹

$$\begin{aligned} S_1 &= \frac{\epsilon_{\mu\nu\rho}}{|x_{12}||x_{23}|^3|x_{31}|} \left[x_{23}^\mu x_{12}^\nu \varepsilon_2^\rho \vec{\varepsilon}_3 \cdot \vec{x}_{23} - \frac{1}{2}(|x_{12}|^2 x_{23}^\mu + |x_{23}|^2 x_{12}^\mu) \varepsilon_2^\nu \varepsilon_3^\rho \right], \\ S_2 &= \frac{\epsilon_{\mu\nu\rho}}{|x_{12}||x_{23}||x_{31}|^3} \left[x_{31}^\mu x_{23}^\nu \varepsilon_3^\rho \vec{\varepsilon}_1 \cdot \vec{x}_{31} - \frac{1}{2}(|x_{23}|^2 x_{31}^\mu + |x_{31}|^2 x_{23}^\mu) \varepsilon_3^\nu \varepsilon_1^\rho \right], \\ S_3 &= \frac{\epsilon_{\mu\nu\rho}}{|x_{12}|^3|x_{23}||x_{31}|} \left[x_{12}^\mu x_{31}^\nu \varepsilon_1^\rho \vec{\varepsilon}_2 \cdot \vec{x}_{12} - \frac{1}{2}(|x_{31}|^2 x_{12}^\mu + |x_{12}|^2 x_{31}^\mu) \varepsilon_1^\nu \varepsilon_2^\rho \right]. \end{aligned} \quad (2.15)$$

We may also write them in terms of polarization spinors as

$$\begin{aligned} S_1 &= \frac{i}{4} \frac{|x_{12}|}{|x_{23}||x_{31}|} [(\lambda_2 \tilde{\mathbf{x}}_{12} \tilde{\mathbf{x}}_{23} \lambda_2)(\lambda_3 \mathbf{x}_{23} \lambda_3) + (\lambda_2 \lambda_3)(\lambda_2(\tilde{\mathbf{x}}_{23} + \tilde{\mathbf{x}}_{12})\lambda_3)] \\ &= \frac{i}{4} \frac{1}{|x_{12}||x_{23}||x_{31}|} (\lambda_3 \mathbf{x}_{31} \mathbf{x}_{12} \lambda_2) (\lambda_2 \tilde{\mathbf{x}}_{23} \lambda_3), \\ S_2 &= \frac{i}{4} \frac{1}{|x_{12}||x_{23}||x_{31}|} (\lambda_1 \mathbf{x}_{12} \mathbf{x}_{23} \lambda_3) (\lambda_3 \tilde{\mathbf{x}}_{31} \lambda_1), \\ S_3 &= \frac{i}{4} \frac{1}{|x_{12}||x_{23}||x_{31}|} (\lambda_2 \mathbf{x}_{23} \mathbf{x}_{31} \lambda_1) (\lambda_1 \tilde{\mathbf{x}}_{12} \lambda_2). \end{aligned} \quad (2.16)$$

Like the (P_i, Q_i) defined above, these structures are Poincaré invariant and homogeneous of degree zero under D , and moreover it can be checked that they transform under the inversion R into minus themselves. Therefore they are conformally invariant structures.² In fact, S_1 for instance is proportional to the correlator of a scalar operator at x_1 and two spin-1 currents at x_2 and x_3 contracted with polarization vectors ε_2 and ε_3 in the theory of a free massless fermion

$$\langle (\bar{\psi}\psi)(x_1) (\bar{\psi}\varepsilon_2 \cdot \gamma\psi)(x_2) (\bar{\psi}\varepsilon_3 \cdot \gamma\psi)(x_3) \rangle_{\text{free}} = \frac{c}{x_{12}^2 x_{31}^2} S_1, \quad (2.17)$$

where c is some normalization constant.

By the counting of degrees of freedom given above, we expect that there should be some non-linear relations between the S_i 's and the P_i, Q_i 's, that are even in the S_i 's.

¹We thank J. Penedones for a very useful discussion.

²Recall that the special conformal generators may be written in terms of translations and inversion as $K_\mu = R P_\mu R$.

This is intuitively clear, since a product of two Levi-Civita tensors can be expressed in terms of the metric. Indeed, one finds the following explicit relations

$$\begin{aligned}
S_3^2 &= \frac{1}{16}P_3^2(Q_1Q_2 - P_3^2), & S_1^2 &= \frac{1}{16}P_1^2(Q_2Q_3 - P_1^2), & S_2^2 &= \frac{1}{16}P_2^2(Q_3Q_1 - P_2^2), \\
S_3S_1 &= \frac{1}{16}P_1P_3(P_1P_3 - P_2Q_2), & S_2S_3 &= \frac{1}{16}P_2P_3(P_2P_3 - P_1Q_1), \\
S_1S_2 &= \frac{1}{16}P_1P_2(P_1P_2 - P_3Q_3).
\end{aligned} \tag{2.18}$$

Therefore, while one still needs these S_i 's to build the general conformally invariant 3-point function, it is sufficient to consider structures which are at most linear in S_i . However, we observe that the counting of independent tensor structures in the 3-point functions is further complicated by the existence of the following additional identities

$$\begin{aligned}
(P_1^2Q_1 - P_2^2Q_2)S_3 + (P_3^2 - Q_1Q_2)(Q_1S_1 - Q_2S_2) &= 0 \\
(P_1^2Q_1 - P_3^2Q_3)S_2 + (P_2^2 - Q_1Q_3)(Q_1S_1 - Q_3S_3) &= 0 \\
(P_2^2Q_2 - P_3^2Q_3)S_1 + (P_1^2 - Q_2Q_3)(Q_2S_2 - Q_3S_3) &= 0.
\end{aligned} \tag{2.19}$$

These can be seen to be true by squaring the left-hand side, using (2.18) and then (2.14). Note that these are not independent conditions, since any one of them follows from the other two. Such relations, together with (2.14), must be taken into account to write down the linearly independent structures which contribute to a given 3-point function.

To summarize, we then conclude that the most general expression for the three point function $F(\{x_i, \lambda_i, s_i\})$ consistent with conformal symmetry can be written as

$$F(\{x_i, \lambda_i, s_i\}) = \frac{1}{|x_{12}||x_{23}||x_{31}|} G_{(s_1, s_2, s_3)}(P_i, Q_i, S_i) \tag{2.20}$$

where $G_{(s_1, s_2, s_3)}$ is a polynomial in (P_i, Q_i, S_i) which is at most linear in S_i and is homogeneous of degree $2s_i$ in λ_i .

2.2 Some simple examples

Let us now discuss a few specific examples. For simplicity we will mostly concentrate on the case in which the currents are abelian, so we impose symmetry under pairwise exchanges when some of the spins are equal. It is straightforward to drop this assumptions and work with more general non-abelian currents.

First consider the $\langle J_1 J_1 J_1 \rangle$ 3-point function. It is easy to see that if there is only a $U(1)$ vector current in the theory, then this correlator is trivial by imposing

symmetry under exchange of a pair of currents.³ Indeed, while there are potential conformal structures with the right degree, such as $P_1 P_2 P_3$, $Q_1 Q_2 Q_3$, $S_1 Q_1$, $S_2 Q_2$, $S_3 Q_3$, it is not possible to build out of them a structure invariant under the exchanges $(x_1, \lambda_1) \leftrightarrow (x_2, \lambda_2)$, $(x_1, \lambda_1) \leftrightarrow (x_3, \lambda_3)$, $(x_3, \lambda_3) \leftrightarrow (x_2, \lambda_2)$. For example, notice that under $(x_1, \lambda_1) \leftrightarrow (x_2, \lambda_2)$, the basic structures transform as

$$\begin{aligned} P_1 &\rightarrow -P_2, & P_2 &\rightarrow -P_1, & P_3 &\rightarrow -P_3, & Q_1 &\rightarrow -Q_2, & Q_2 &\rightarrow -Q_1, & Q_3 &\rightarrow -Q_3 \\ S_1 &\rightarrow S_2, & S_2 &\rightarrow S_1, & S_3 &\rightarrow S_3. \end{aligned} \quad (2.21)$$

As a next example, consider $\langle J_2 J_1 J_1 \rangle$. In this case one finds 8 possible tensor structures symmetric under $(x_2, \lambda_2) \leftrightarrow (x_3, \lambda_3)$, 5 of them being parity even and 3 parity odd. Upon using the relations (2.14) and (2.19), these can be reduced to the following 6 linearly independent structures⁴

$$\begin{aligned} \langle J_2 J_1 J_1 \rangle : & \quad P_1^2 Q_1^2 \quad P_2^2 P_3^2 \quad Q_1^2 Q_2 Q_3 \quad P_1 P_2 P_3 Q_1 \\ & \quad Q_1^2 S_1 \quad P_2^2 S_3 + P_3^2 S_2. \end{aligned} \quad (2.22)$$

Imposing that J_2 and J_1 are conserved further restricts the possible independent tensor structures. This is explained in the next section. Here we list some further examples of the structures one gets by imposing conformal invariance alone.

Consider now $\langle J_2 J_2 J_1 \rangle$. One can see from the generating functions that this correlator vanishes in the free scalar and free fermion theory. This is, in fact, true in general if J_2 and J_1 are conserved abelian currents, see next section. Without imposing conservation, the independent conformal invariants symmetric under $(x_1, \lambda_1) \leftrightarrow (x_2, \lambda_2)$ are in this case the following 4 structures (here we used (2.19) to eliminate one linearly dependent structure)

$$\begin{aligned} \langle J_2 J_2 J_1 \rangle : & \quad Q_1 Q_2 (P_1^2 Q_1 - P_2^2 Q_2) \quad P_3^2 (P_1^2 Q_1 - P_2^2 Q_2) \\ & \quad S_3 (P_1^2 Q_1 - P_2^2 Q_2) \quad P_3^2 (S_1 Q_1 - S_2 Q_2). \end{aligned} \quad (2.23)$$

Let us analyze next the 3-point function of the stress energy tensor, $\langle J_2 J_2 J_2 \rangle$. In this case one finds 7 structures which do not involve the parity odd S_i 's, plus 5 more structures linear in S_i . Using the cubic relation (2.14) and the identities (2.19), these can be reduced to the following independent 4 parity even plus 2 parity odd structures

$$\begin{aligned} \langle J_2 J_2 J_2 \rangle : & \quad P_1^2 P_2^2 P_3^2 \quad Q_1^2 Q_2^2 Q_3^2 \quad P_1 P_2 P_3 Q_1 Q_2 Q_3 \quad P_1^4 Q_1^2 + P_2^4 Q_2^2 + P_3^4 Q_3^2 \\ & \quad P_1^2 Q_1^2 S_1 + P_2^2 Q_2^2 S_2 + P_3^2 Q_3^2 S_3 \quad P_1^2 P_2^2 S_3 + P_2^2 P_3^2 S_1 + P_3^2 P_1^2 S_2. \end{aligned} \quad (2.24)$$

³In the case of a non-abelian spin 1 current J_1^a , the 3-point function $\langle J_1^a J_1^b J_1^c \rangle$ is non-trivial and proportional to the structure constants f^{abc} [8]. The parity even conserved structures can be obtained from free theories, see the generating function (3.6) below. Also in this case we find a new parity violating term which was not given in [8], see the Appendix.

⁴To reduce the number of parity odd structures, note that (2.14) and (2.19) can be shown to imply the identity $Q_1 Q_2 S_2 + Q_1 Q_3 S_3 - 2P_2^2 S_3 - 2P_3^2 S_2 - Q_1^2 S_1 = 0$ and its cyclically related analogs.

Let us now look at some higher spin example. Consider for example the case $\langle J_4 J_1 J_1 \rangle$. In this case, we find the following 4+2 independent conformal invariant structures (again, we need to use (2.14) and (2.19) to eliminate some linearly dependent structures)

$$\langle J_4 J_1 J_1 \rangle : \begin{array}{cccc} P_1^2 Q_1^4 & P_2^2 P_3^2 Q_1^2 & Q_1^4 Q_2 Q_3 & P_1 P_2 P_3 Q_1^3 \\ Q_1^4 S_1 & P_2^2 Q_1^2 S_3 + P_3^2 Q_1^2 S_2. \end{array} \quad (2.25)$$

Note that these structures are the same as the ones for $\langle J_2 J_1 J_1 \rangle$ in (2.22), except that they are multiplied by Q_1^2 . It is not difficult to see that this pattern persists for $\langle J_s J_1 J_1 \rangle$, with s arbitrary even spin, namely one gets the same set of structures as $\langle J_2 J_1 J_1 \rangle$ multiplied by Q_1^{s-2} . For s odd, one can also see that the number of independent structures in $\langle J_s J_1 J_1 \rangle$ does not increase with s . In this case, the conformally invariant structures symmetric under the exchange $(x_2, \lambda_2) \leftrightarrow (x_3, \lambda_3)$ are

$$\langle J_s J_1 J_1 \rangle : \begin{array}{cc} Q_1^{s-1} (P_2^2 Q_2 - P_3^2 Q_3) & s = 3, 5, 7, \dots \\ Q_1^{s-2} (P_3^2 S_2 - P_2^2 S_3) & Q_1^{s-1} (Q_3 S_3 - Q_2 S_2) \end{array} \quad (2.26)$$

As a last more complicated example we can consider $\langle J_4 J_2 J_1 \rangle$. In this case we find the following 15 independent conformal invariant structures

$$\langle J_4 J_2 J_1 \rangle : \begin{array}{cccc} Q_1^4 Q_2^2 Q_3 & Q_1^3 Q_2 Q_3 P_3^2 & Q_1^2 Q_3 P_3^4 & Q_1^3 Q_2^2 P_2^2 \\ Q_1^2 Q_2 P_2^2 P_3^2 & Q_1 P_2^2 P_3^4 & Q_1^4 Q_2 P_1^2 & Q_1^3 P_1^2 P_3^2 \\ Q_1^3 P_1^2 S_3 & Q_1^4 Q_2 S_1 & Q_1^3 P_3^2 S_1 & Q_1^3 Q_2^2 S_2 \\ Q_1^2 Q_2 S_2 & Q_1 P_2^2 P_3^2 S_3 & Q_1 P_3^4 S_2. \end{array} \quad (2.27)$$

One can proceed analogously to determine the allowed tensor structures for further examples of 3-point functions. We will not write here explicitly the list of conformally invariant tensor structures in other cases, which rapidly increases with the spins, but in the next section we will give the final results after imposing current conservation. Some examples involving the scalar operators also will be presented in section 3.3.

3 Three point functions of conserved higher spin currents

3.1 Summary of results

Now, we want to investigate the consequence of imposing the current conservation condition $\partial^\nu J_{\nu\mu_1\cdots\mu_{s-1}} = 0$. In terms of the spinorial generating function, this amounts to

$$(\partial_\lambda \not{\partial}_x \partial_\lambda) J_s(x, \lambda) = 0. \quad (3.1)$$

If we impose the conservation law on the current J_{s_i} , then the correlation function obeys (up to contact terms which we neglect)

$$(\partial_{\lambda_i} \not{x}_i \partial_{\lambda_i}) F(\{x_j, \lambda_j, s_j\}) = 0. \quad (3.2)$$

After expressing F in terms of P_i, Q_i, S_i as in the previous section, it is unclear to us how to solve the current conservation condition in general. Nevertheless, it is straightforward to examine this case by case. This can be done as follows. First, we write a given 3-point function as a linear combination of conformally invariant independent structures, as explained in the previous section

$$F(\{x_j, \lambda_j, s_j\}) = \sum_k a_k \mathcal{T}_k(P_i, Q_i, S_i), \quad (3.3)$$

where \mathcal{T}_k denote a basis of linearly independent tensor structures. After evaluating (3.2), it is convenient to use conformal invariance to set e.g. $x_1 = -x_2 = (1, 0, 0)$, $x_3 = (0, 0, 0)$. Expanding the equation (3.2) in powers of polarization spinors and requiring that each monomial vanishes, we get a homogeneous linear system of equations for the coefficients a_k . Its non-trivial solutions (if any) determine the independent conserved tensor structures for the given 3-point function. Our main results are summarized below.

The three point function of *conserved* currents J_s of spin s in a three dimensional CFT have the following structure

$$\langle J_{s_1} J_{s_2} J_{s_3} \rangle = a_1 \langle J_{s_1} J_{s_2} J_{s_3} \rangle_B + a_2 \langle J_{s_1} J_{s_2} J_{s_3} \rangle_F + b \langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\text{odd}}. \quad (3.4)$$

Here $\langle JJJ \rangle_B$, $\langle JJJ \rangle_F$ preserve parity, whereas $\langle JJJ \rangle_{\text{odd}}$ violates parity. a_1, a_2 and b are constant coefficients which depend on the particular set of currents appearing on the RHS. $\langle JJJ \rangle_B$ and $\langle JJJ \rangle_F$ are given in terms of the generating functions

$$\begin{aligned} \mathcal{F}_B &= \frac{1}{|x_{12}| |x_{23}| |x_{31}|} \exp \left[\frac{1}{2} (Q_1 + Q_2 + Q_3) \right] \cosh P_1 \cosh P_2 \cosh P_3, \\ \mathcal{F}_F &= \frac{1}{|x_{12}| |x_{23}| |x_{31}|} \exp \left[\frac{1}{2} (Q_1 + Q_2 + Q_3) \right] \sinh P_1 \sinh P_2 \sinh P_3, \end{aligned} \quad (3.5)$$

expanded to order $\lambda_1^{2s_1} \lambda_2^{2s_2} \lambda_3^{2s_3}$. The structure $\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\text{odd}}$, on the other hand, is found to be unique when the three spins obey triangular inequality, namely $s_1 \leq s_2 + s_3$, $s_2 \leq s_3 + s_1$, $s_3 \leq s_1 + s_2$, and vanishes when the triangular inequality is violated. Note that this structure holds for general nonabelian higher spin currents, and does not require any assumption on symmetries among currents of the same spin. In the case where there is only one current of a certain spin, for instance, some of these structures are disallowed by the symmetry that exchanges two currents in the correlator.

When $s_1 + s_2 + s_3 = \text{even}$, $\langle JJJ \rangle_B$ and $\langle JJJ \rangle_F$ are proportional to the corresponding three point function of currents in the free theory of a massless scalar and that of a massless fermion. When $s_1 + s_2 + s_3 = \text{odd}$, the structures $\langle JJJ \rangle_B$ and $\langle JJJ \rangle_F$ do not show up in the free theory of a single scalar or a single fermion, but do appear in correlators of nonabelian currents in the free theories of several scalars or several fermions. For instance, suppose we take NM free complex massless scalars or fermions, and consider $U(N)$ invariant higher spin currents J_s^a , which are bilinears in the fields and take value in the global $U(M)$ flavor symmetry algebra. Their three point functions are given in terms of the generating functions⁵

$$\begin{aligned}\mathcal{F}_{boson}^{abc} &= \frac{d^{abc} \cosh \left[\frac{1}{2}(Q_1 + Q_2 + Q_3) \right] + f^{abc} \sinh \left[\frac{1}{2}(Q_1 + Q_2 + Q_3) \right]}{|x_{12}||x_{23}||x_{31}|} \cosh P_1 \cosh P_2 \cosh P_3, \\ \mathcal{F}_{fermion}^{abc} &= \frac{d^{abc} \sinh \left[\frac{1}{2}(Q_1 + Q_2 + Q_3) \right] + f^{abc} \cosh \left[\frac{1}{2}(Q_1 + Q_2 + Q_3) \right]}{|x_{12}||x_{23}||x_{31}|} \sinh P_1 \sinh P_2 \sinh P_3.\end{aligned}\tag{3.6}$$

Here $d^{abc} = \text{Tr}(T^a T^{(b} T^{c)})$, $f^{abc} = \text{Tr}(T^a T^{[b} T^{c]})$, where T^a are the generators of $U(M)$ flavor symmetry.

While we do not know a closed form formula for $\langle J_{s_1} J_{s_2} J_{s_3} \rangle_{\text{odd}}$ with general spins, in all the cases we have examined, we find that such a structure exists and is unique when the triangular inequalities on the spins are obeyed, and does not exist when triangular inequalities are violated. We conjecture that this statement is true for all spins. In the next subsection we will present some explicit formulae in special cases.

The $\langle JJJ \rangle_{\text{odd}}$ structures do not show up in free field theories. It can arise in interacting three dimensional CFTs that violate parity. For example it is non-zero at odd loop order in the theory of a fundamental fermion coupled to a $U(N)$ Chern-Simons gauge field, as will be demonstrated in [10].

3.2 Examples

Let us start from the $\langle J_2 J_1 J_1 \rangle$ example discussed above. We write the correlator as a linear combination of the 6 conformally invariant structures given in (2.22)

$$\begin{aligned}\langle J_2 J_1 J_1 \rangle &= \frac{1}{|x_{12}||x_{23}||x_{31}|} [a_1 P_1^2 Q_1^2 + a_2 P_2^2 P_3^2 + a_3 Q_1^2 Q_2 Q_3 + a_4 P_1 P_2 P_3 Q_1 \\ &\quad b_1 Q_1^2 S_1 + b_2 (P_2^2 S_3 + P_3^2 S_2)]\end{aligned}\tag{3.7}$$

Imposing conservation one finds the conditions $a_2 = -4a_1$, $a_3 = -5/2a_1$ and $b_2 = 2b_1$, so the end result is that conformal symmetry and current conservation restrict the

⁵This can be derived analogously to the appendix of [6].

3-point function to the form

$$\langle J_2 J_1 J_1 \rangle = \frac{1}{|x_{12}||x_{23}||x_{31}|} \left[a_1 \left(P_1^2 Q_1^2 - 4P_2^2 P_3^2 - \frac{5}{2} Q_1^2 Q_2 Q_3 \right) + a_4 P_1 P_2 P_3 Q_1 \right. \\ \left. + b_1 (Q_1^2 S_1 + 2P_2^2 S_3 + 2P_3^2 S_2) \right] . \quad (3.8)$$

As in [8], the two parity even structures in the first line are linear combinations of free scalar and free fermion structures, see eq. (2.13) (the term proportional to a_4 corresponds to free fermions, while the term proportional to a_1 is a linear combination of free scalars and free fermions). On the other hand, the parity odd structure in the second line was not given in the analysis by Osborn and Petkou [8].

In the case of $\langle J_2 J_2 J_1 \rangle$, we start from a linear combination of the 4 structures found in the previous section

$$\langle J_2 J_2 J_1 \rangle = \frac{1}{|x_{12}||x_{23}||x_{31}|} \left[a_1 Q_1 Q_2 (P_1^2 Q_1 - P_2^2 Q_2) + a_2 P_3^2 (P_1^2 Q_1 - P_2^2 Q_2) \right. \\ \left. + b_1 S_3 (P_1^2 Q_1 - P_2^2 Q_2) + b_2 P_3^2 (S_1 Q_1 - S_2 Q_2) \right] , \quad (3.9)$$

and imposing conservation we find that $a_1 = a_2 = b_1 = b_2 = 0$. Therefore we conclude that the correlator $\langle J_2 J_2 J_1 \rangle$ vanishes in any CFT in which the currents J_2 and J_1 are conserved. Note that here we have assumed the two J_2 's to be the same spin-2 current, and have imposed symmetry under the exchange of the two currents on the RHS. In theories with more than one conserved spin-2 currents, the correlator of two spin-2 and one spin-1 currents can be nontrivial, as discussed in the previous subsection.

Next, let us look at the 3-point function of the stress tensor $T = J_2$. In this case, following the analysis above, we write

$$\langle J_2 J_2 J_2 \rangle = \frac{1}{|x_{12}||x_{23}||x_{31}|} \left[a_1 P_1^2 P_2^2 P_3^2 + a_2 Q_1^2 Q_2^2 Q_3^2 + a_3 P_1 P_2 P_3 Q_1 Q_2 Q_3 \right. \\ \left. + a_4 (P_1^4 Q_1^2 + P_2^4 Q_2^2 + P_3^4 Q_3^2) \right. \\ \left. + b_1 (P_3^2 Q_3^2 S_3 + P_1^2 Q_1^2 S_1 + P_2^2 Q_2^2 S_2) \right. \\ \left. + b_2 (P_1^2 P_2^2 S_3 + P_2^2 P_3^2 S_1 + P_3^2 P_1^2 S_2) \right] . \quad (3.10)$$

The conservation equations now yield the conditions

$$a_3 = \frac{5}{4} a_1 - \frac{16}{5} a_2 , \quad a_4 = -\frac{8}{15} a_2 , \quad b_2 = 5b_1 . \quad (3.11)$$

Therefore we find that there are 2 independent parity even structures, plus one additional parity odd structure. The two parity preserving structures can be seen to be a linear combination of free scalars and free fermions results, which agrees with [8]. The

additional parity odd conserved structure

$$\langle J_2 J_2 J_2 \rangle_{\text{parity odd}} = \frac{(P_1^2 Q_1^2 + 5P_2^2 P_3^2)S_1 + (P_2^2 Q_2^2 + 5P_3^2 P_1^2)S_2 + (P_3^2 Q_3^2 + 5P_1^2 P_2^2)S_3}{|x_{12}||x_{23}||x_{31}|} \quad (3.12)$$

is new, and it can appear for instance in parity violating Chern-Simons matter theories [10].

Next consider the $\langle J_s J_1 J_1 \rangle$ correlator. We should distinguish the cases with s even and s odd. In the former case, we start from (see (2.22) and the following comments)

$$\begin{aligned} \langle J_s J_1 J_1 \rangle = \frac{1}{|x_{12}||x_{23}||x_{31}|} Q_1^{s-2} [a_1 P_1^2 Q_1^2 + a_3 P_2^2 P_3^2 + a_2 Q_1^2 Q_2 Q_3 + a_4 P_1 P_2 P_3 Q_1 \\ + b_1 Q_1^2 S_1 + b_2 (P_2^2 S_3 + P_3^2 S_2)] . \quad s = 2, 4, 6, \dots \end{aligned} \quad (3.13)$$

Imposing conservation on the spin 1 current, we get the constraints

$$\begin{aligned} a_2 = -2sa_1, \quad a_3 = -\frac{2s+1}{2s-2}a_1 \\ b_2 = sb_1 \end{aligned} \quad (3.14)$$

hence

$$\begin{aligned} \langle J_s J_1 J_1 \rangle = \frac{1}{|x_{12}||x_{23}||x_{31}|} Q_1^{s-2} \left[a_1 (P_1^2 Q_1^2 - \frac{2s+1}{2s-2} Q_1^2 Q_2 Q_3 - 2s P_2^2 P_3^2) + a_4 P_1 P_2 P_3 Q_1 \right. \\ \left. + b_1 (Q_1^2 S_1 + s P_2^2 S_3 + s P_3^2 S_2) \right] \quad s = 2, 4, 6, \dots \end{aligned} \quad (3.15)$$

The two independent structures that do not involve S_i are linear combinations of free scalars and free fermions results (the structure proportional to a_4 is precisely the free fermion answer, while the one proportional to a_1 is a linear combination of fermions and bosons results, as can be checked from (2.13)). Notice that even though we have not yet imposed conservation on J_s , this is automatically satisfied by these two structures. On the other hand, the structure which contains S_i in the second line above does not satisfy conservation on J_s for $s \geq 4$ unless $b_1=0$, therefore we conclude that it cannot appear in a theory in which J_s , $s \geq 4$ is exactly conserved.⁶ The case of $\langle J_s J_1 J_1 \rangle$ with odd s can be studied similarly, starting from (2.26). Imposing conservation on J_1 , one finds that the parity even structure in (2.26) is not permitted, and the relative

⁶However, it is expected that this structure can appear for example at odd loop order in the theory of a fundamental fermion coupled to CS gauge field. For $s = 4$, this structure can be related to an anomalous Ward identity obeyed by the 3-point function $\langle J_4 J_1 J_1 \rangle$, which follows from the fact that the divergence of J_4 mixes with “multi-trace” operators, including in particular $J_1 J_1$ [10].

coefficients of the two parity odd structures is fixed as

$$\langle J_s J_1 J_1 \rangle = \frac{b}{|x_{12}||x_{23}||x_{31}|} Q_1^{s-2} \left[(Q_1 Q_2 + s P_3^2) S_2 - (Q_1 Q_3 + s P_2^2) S_3 \right] \quad s = 3, 5, \dots \quad (3.16)$$

Note that since this is a parity breaking structure, it does not arise in particular in free theories. If one further imposes conservation on J_s , one finds that it is only possible if $b = 0$, hence this 3-point function vanishes in a 3d CFT with conserved J_s (s odd).

For $\langle J_4 J_2 J_1 \rangle$, starting from a linear combination of the 15 conformally invariant structures listed in (2.27) and imposing conservation on all currents, we end up with the following independent conserved structures⁷

$$\begin{aligned} \langle J_4 J_2 J_1 \rangle = & \frac{1}{|x_{12}||x_{23}||x_{31}|} \left[a_1 \left(Q_1^4 Q_2^2 Q_3 - \frac{64}{15} Q_1 P_2^2 P_3^4 - \frac{64}{15} Q_1^3 P_1^2 P_3^2 - \frac{128}{15} Q_1^2 Q_2 P_2^2 P_3^2 \right. \right. \\ & \left. \left. - \frac{4}{3} Q_1^4 Q_2 P_1^2 - \frac{8}{5} Q_1^3 Q_2^2 P_2^2 - \frac{16}{5} Q_1^2 Q_3 P_3^4 \right) + a_2 P_1 P_2 P_3 Q_1^2 (Q_1 Q_2 + 2 P_3^2) \right]. \end{aligned} \quad (3.17)$$

Note that in this case all the parity odd structures in (2.27) are killed by imposing conservation on the currents.⁸ This is in agreement with the claim in the previous subsection that the parity odd structure does not exist when the triangular inequalities on the spins are violated. Observe that the two conserved structures found above can be obtained by expanding to the appropriate powers of polarization spinors the two following objects

$$\begin{aligned} & \frac{1}{|x_{12}||x_{23}||x_{31}|} \sinh \left[\frac{1}{2} (Q_1 + Q_2 + Q_3) \right] \cosh P_1 \cosh P_2 \cosh P_3 \\ & \frac{1}{|x_{12}||x_{23}||x_{31}|} \cosh \left[\frac{1}{2} (Q_1 + Q_2 + Q_3) \right] \sinh P_1 \sinh P_2 \sinh P_3, \end{aligned} \quad (3.18)$$

which are just the terms proportional to f^{abc} in the generating functions of 3-point correlators of non-abelian currents in free theories, see eq. (3.6). In particular expanding the second line to order $\lambda_1^8 \lambda_2^4 \lambda_3^2$ one gets the structure proportional to a_2 in (3.17), while expanding the first line gives a linear combination of the a_1 and a_2 structures in (3.17). It is easy to verify that the expansion of (3.18) to the power $\lambda_1^{2s_3} \lambda_2^{2s_2} \lambda_3^{2s_3}$ generates new non-trivial conformal invariant and conserved structures in the case $\sum s_i = \text{odd}$. Note however that (3.18) are antisymmetric under any pairwise interchange $(x_i, \lambda_i) \leftrightarrow (x_j, \lambda_j)$, hence they cannot be used to generate 3-point structures for abelian currents when any two of the spins are equal. So our conclusion, as summarized in Section 3.1, is that for $\langle J_{s_1} J_{s_2} J_{s_3} \rangle$ with $\sum s_i = \text{odd}$ and $s_1 \neq s_2 \neq s_3$ there

⁷Here we have used (2.14) to rewrite the tensor structures in a simpler form.

⁸Two independent linear combinations of the parity odd structures in (2.27) would be allowed if one imposes conservation only on J_2 and J_1 .

are two parity even conformally invariant and conserved structures besides those of a free massless scalar/fermion. By looking at several further examples, it appears that (2.13) and (3.18) exhaust all possible parity even conserved structures.

We end this section by listing the results for the correlators $\langle J_4 J_2 J_2 \rangle$, $\langle J_4 J_4 J_2 \rangle$ and $\langle J_4 J_4 J_4 \rangle$, obtained after imposing conformal invariance together with conservation on all currents. We find

$$\begin{aligned} \langle J_4 J_2 J_2 \rangle &= a_1 \langle J_4 J_2 J_2 \rangle_{\text{free scalars}} + a_2 \langle J_4 J_2 J_2 \rangle_{\text{free fermions}} \\ &+ b_1 \frac{Q_1}{|x_{12}||x_{23}||x_{31}|} \left[2S_1 Q_1 (P_1 P_2 P_3 Q_1 + P_1^2 Q_1^2 - 6P_2^2 P_3^2) \right. \\ &\quad \left. - (S_2 Q_2 + S_3 Q_3) (6P_2^2 P_3^2 + 5Q_1^2 Q_2 Q_3) \right]. \end{aligned} \quad (3.19)$$

$$\begin{aligned} \langle J_4 J_4 J_2 \rangle &= a_1 \langle J_4 J_4 J_2 \rangle_{\text{free scalars}} + a_2 \langle J_4 J_4 J_2 \rangle_{\text{free fermions}} \\ &+ b_1 \frac{1}{|x_{12}||x_{23}||x_{31}|} \left[Q_1^4 Q_2^3 Q_3 S_1 + Q_1^3 Q_2^4 Q_3 S_2 + \frac{1}{10} \left(126 P_1^3 P_2 P_3 Q_1^2 Q_2 \right. \right. \\ &\quad \left. - 14 P_1^4 Q_1^2 (P_3^2 + 3Q_1 Q_2) + P_1^2 Q_1 Q_3 (24 P_3^4 - 37 P_3^2 Q_1 Q_2 - 117 Q_1^2 Q_2^2) \right. \\ &\quad \left. - \frac{1}{2} Q_3^2 (2P_3^6 + 67 P_3^4 Q_1 Q_2 - 10 P_3^2 Q_1^2 Q_2^2 - 189 Q_1^3 Q_2^3) + (x_1, \lambda_1) \leftrightarrow (x_2, \lambda_2) \right) S_3 \Big]. \end{aligned} \quad (3.20)$$

$$\begin{aligned} \langle J_4 J_4 J_4 \rangle &= a_1 \langle J_4 J_4 J_4 \rangle_{\text{free scalars}} + a_2 \langle J_4 J_4 J_4 \rangle_{\text{free fermions}} \\ &+ b_1 \frac{1}{|x_{12}||x_{23}||x_{31}|} \left[Q_1 S_1 \left(16 P_1^6 Q_1^3 + 128 P_2^6 Q_2^3 + 128 P_3^6 Q_3^3 + 8 P_1^4 Q_1^3 Q_2 Q_3 \right. \right. \\ &\quad \left. - 608 P_2^4 Q_1 Q_2^3 Q_3 - 608 P_3^4 Q_1 Q_2 Q_3^3 - 414 P_1^2 Q_1^3 Q_2^2 Q_3^2 + 627 Q_1^3 Q_2^3 Q_3^3 \right) + \text{cyclic} \Big]. \end{aligned} \quad (3.21)$$

3.3 Examples involving a scalar operator

The explicit examples discussed so far involve currents of spins $s_i \geq 1$. Of course, the formalism we set up above can be easily applied to the case in which the 3-point functions involve scalar primary operators. Let us imagine that the CFT under study has a scalar primary \mathcal{O}_Δ of dimension Δ . In the specific example of the duality between Vasiliev's theories [7] and vector models [3][4], the relevant case is $\Delta = 1$ or $\Delta = 2$, and the scalar operator can be either even or odd under parity.

The simplest case to consider is the 3-point function involving a higher spin current and two scalars, $\langle J_s(x_1, \lambda_1) \mathcal{O}_\Delta(x_2) \mathcal{O}_\Delta(x_3) \rangle$. In this case it is easy to see that among the conformally invariant tensor structures (P_i, Q_i, S_i) we can only use Q_1 . Therefore we immediately conclude that conformal invariance implies

$$\langle J_s(x_1, \lambda_1) \mathcal{O}_\Delta(x_2) \mathcal{O}_\Delta(x_3) \rangle = \frac{a}{|x_{12}||x_{23}|^{2\Delta-1}|x_{31}|} (Q_1)^s, \quad s = 2, 4, 6, \dots \quad (3.22)$$

while it vanishes if s is odd (this is because Q_1 changes sign if we exchange x_2 and x_3).

Here a is some arbitrary constant. One can see that this obeys the Ward identity for current conservation on J_s for any choice of Δ .

In the case of the correlator of two higher spin currents at x_1, x_2 and one scalar at x_3 , we see that the possible building blocks are Q_1, Q_2, P_3, S_3 . Then we have

$$\langle J_{s_1}(x_1, \lambda_1) J_{s_2}(x_2, \lambda_2) \mathcal{O}_\Delta(x_3) \rangle = \frac{1}{|x_{12}|^{2-\Delta} |x_{23}|^\Delta |x_{31}|^\Delta} \left[G(Q_1, Q_2, P_3) + S_3 \tilde{G}(Q_1, Q_2, P_3) \right], \quad (3.23)$$

where G and \tilde{G} are homogeneous of degree respectively $(2s_1, 2s_2)$ and $(2s_1 - 2, 2s_2 - 2)$ in (λ_1, λ_2) . In a parity invariant CFT, either G or \tilde{G} must vanish. If \mathcal{O}_Δ is parity even, then \tilde{G} should vanish and conversely G vanishes if \mathcal{O}_Δ is parity odd. More generally, both G and \tilde{G} can appear if parity is broken. For example, in the free massless scalar theory with $\Delta = 1$ parity even scalar, $\tilde{G}=0$ and the generating function for $\langle J_{s_1} J_{s_2} J_0 \rangle$ is given by (2.13) upon setting $\lambda_3 = 0$. In the free fermion theory, with $\Delta = 2$ parity odd scalar operator, it may be shown that a generating function for $\langle J_{s_1} J_{s_2} J_0 \rangle$ of all spins can be written as⁹

$$\mathcal{F}_{\langle J_{s_1} J_{s_2} J_0 \rangle}^{\text{fermi}} = \frac{1}{|x_{23}|^2 |x_{31}|^2} S_3 \cosh \left(\frac{Q_1 + Q_2}{2} \right) \frac{\sinh P_3}{P_3}. \quad (3.24)$$

In particular for $\langle J_1 J_1 J_0 \rangle$ this reproduces the result quoted earlier.

It is straightforward to work out some specific examples. For instance, before imposing conservation, we have

$$\langle J_1(x_1, \lambda_1) J_1(x_2, \lambda_2) \mathcal{O}_\Delta(x_3) \rangle = \frac{1}{|x_{12}|^{2-\Delta} |x_{23}|^\Delta |x_{31}|^\Delta} [a_1 Q_1 Q_2 + a_2 P_3^2 + b S_3]. \quad (3.25)$$

Requiring that J_1 is a conserved current yields the condition $(4 - 2\Delta)a_1 - \Delta a_2 = 0$ and b arbitrary, therefore we find

$$\langle J_1 J_1 \mathcal{O}_\Delta \rangle = \frac{1}{|x_{12}|^{2-\Delta} |x_{23}|^\Delta |x_{31}|^\Delta} [a (\Delta Q_1 Q_2 + (4 - 2\Delta) P_3^2) + b S_3]. \quad (3.26)$$

For $\Delta = 1$ and $b = 0$ we recognize the free scalar result in eq. (2.13), while for $\Delta = 2$ and $a = 0$ we find the free fermion result, see eq. (2.17). In the case of the critical scalar $U(N)$ theory, where $\Delta = 2 + \mathcal{O}(1/N)$ and \mathcal{O}_Δ is parity even, we should find $\langle J_1 J_1 \mathcal{O}_{\Delta=2} \rangle = a Q_1 Q_2 / (x_{23}^2 x_{31}^2)$. In parity violating Chern-Simons matter theories, we expect that a can be non-zero at odd loop order and b at even loop order.

Going up in spin, consider $\langle J_2 J_1 \mathcal{O}_\Delta \rangle$. In this case we find that conservation on J_2 and J_1 implies that this correlator must vanish, except in the special cases $\Delta = 1$ and

⁹We obtained this simple generating function from Vasiliev's higher spin gauge theory in AdS_4 . Its derivation will be presented elsewhere.

$\Delta = 2$, where we find the structures

$$\begin{aligned}\langle J_2 J_1 \mathcal{O}_{\Delta=1} \rangle &= \frac{a}{|x_{12}| |x_{23}| |x_{31}|} (Q_1^2 Q_2 + 4 Q_1 P_3^2) \\ \langle J_2 J_1 \mathcal{O}_{\Delta=2} \rangle &= \frac{b}{x_{23}^2 x_{31}^2} Q_1 S_3.\end{aligned}\tag{3.27}$$

which are respectively parity even and parity odd. In the free theories of a single scalar or fermion, $\langle J_2 J_1 \mathcal{O}_{\Delta} \rangle$ (with $\Delta = 1, 2$ respectively) vanishes, as can be checked by direct computation or from the generating functions given earlier. However, note that the structure we found for $\langle J_2 J_1 \mathcal{O}_{\Delta=1} \rangle$ is contained in the term proportional to f^{abc} of the boson generating function in eq. (3.6). The $\Delta = 2$ parity odd structure also corresponds to the three point function of non-abelian currents in the theory of several free fermions.

The next non-trivial example is $\langle J_2 J_2 \mathcal{O}_{\Delta} \rangle$, where we find two independent conserved structures

$$\begin{aligned}\langle J_2 J_2 \mathcal{O}_{\Delta} \rangle &= \frac{1}{|x_{12}|^{2-\Delta} |x_{23}|^{\Delta} |x_{31}|^{\Delta}} \left[a \left(\Delta(\Delta+2) Q_1^2 Q_2^2 + \Delta(32-8\Delta) Q_1 Q_2 P_3^2 \right. \right. \\ &\quad \left. \left. + 8(6-6\Delta+\Delta^2) P_3^4 \right) + b S_3 ((\Delta+1) Q_1 Q_2 + (6-2\Delta) P_3^2) \right].\end{aligned}\tag{3.28}$$

One may check that for $\Delta = 1$, $b = 0$ and $\Delta = 2$, $a = 0$ we recover respectively the free scalar and free fermion results, see eq. (2.13) and (3.24).

Let us finally list some examples involving higher spins, namely $\langle J_3 J_1 \mathcal{O}_{\Delta} \rangle$, $\langle J_5 J_1 \mathcal{O}_{\Delta} \rangle$ and $\langle J_4 J_2 \mathcal{O}_{\Delta} \rangle$. Imposing conservation only on J_1 , we find that

$$\begin{aligned}\langle J_3 J_1 \mathcal{O}_{\Delta} \rangle &= \frac{1}{|x_{12}|^{2-\Delta} |x_{23}|^{\Delta} |x_{31}|^{\Delta}} \left[a_3 (\Delta Q_1^3 Q_2 + (8-2\Delta) Q_1^2 P_3^2) + b_3 Q_1^2 S_3 \right], \\ \langle J_5 J_1 \mathcal{O}_{\Delta} \rangle &= \frac{1}{|x_{12}|^{2-\Delta} |x_{23}|^{\Delta} |x_{31}|^{\Delta}} \left[a_5 (\Delta Q_1^5 Q_2 + (12-2\Delta) Q_1^4 P_3^2) + b_5 Q_1^4 S_3 \right].\end{aligned}\tag{3.29}$$

Exact conservation of J_3 then implies

$$\begin{aligned}\langle \partial \cdot J_3 J_1 \mathcal{O}_{\Delta} \rangle = 0 &\leftrightarrow \begin{aligned} b_3 &= 0, \quad \Delta = 1 \\ a_3 &= 0, \quad \Delta = 2, \\ b_3 &= a_3 = 0, \quad \text{if } \Delta \neq 1, 2 \end{aligned}\end{aligned}\tag{3.30}$$

and imposing conservation on J_5 for $\langle J_5 J_1 \mathcal{O}_{\Delta} \rangle$ one finds exactly the same conditions on a_5 , b_5 . The structures which are conserved on all currents just correspond to free scalars and free fermions. Analogous results can be written down for $\langle J_s J_1 \mathcal{O}_{\Delta} \rangle$, $s = 7, 9, \dots$, with the total number of conserved structures not increasing with s .

Finally, for $\langle J_4 J_2 \mathcal{O}_\Delta \rangle$ (similar results apply to $\langle J_s J_2 \mathcal{O}_\Delta \rangle$ for $s \geq 4$), conservation of the stress-tensor J_2 implies

$$\begin{aligned} \langle J_4 J_2 \mathcal{O}_\Delta \rangle = \frac{1}{|x_{12}|^{2-\Delta} |x_{23}|^\Delta |x_{31}|^\Delta} & \left[a \left(\Delta(\Delta+2) Q_1^4 Q_2^2 + 8(15-10\Delta+\Delta^2) Q_1^2 P_3^4 \right. \right. \\ & + 8\Delta(7-\Delta) Q_1^3 Q_2 P_3^2 \\ & \left. \left. + b S_3 Q_1^2 \left((\Delta+1) Q_1 Q_2 + (10-2\Delta) P_3^2 \right) \right] . \end{aligned} \quad (3.31)$$

Analogously as above, conservation on J_4 requires

$$\begin{aligned} \langle \partial \cdot J_4 J_2 \mathcal{O}_\Delta \rangle = 0 \quad \Leftrightarrow \quad & b = 0, \quad \Delta = 1 \\ & a = 0, \quad \Delta = 2, \\ & b = a = 0, \quad \text{if } \Delta \neq 1, 2. \end{aligned} \quad (3.32)$$

It is interesting to observe how imposing conservation of the higher-spin currents singles out the cases $\Delta = 1$, $\Delta = 2$ as special, corresponding to a dual scalar field in AdS_4 with $m^2 = -2$. This is indeed as expected from the structure of the 4d higher spin gauge theory [7].

4 Comments on the holographic dual

In this paper we have found new conformally invariant parity odd tensor structures for three point functions of conserved currents of various spins, and one may ask if they arise in actual three-dimensional CFTs. It will be shown in [10] via explicit perturbative computations that this indeed occurs in parity violating conformal Chern-Simons-matter theories. For now, we will make a few comments on the holographic dual of such CFTs. A large N CFT in three dimensions is expected to be dual to a theory of gravity in AdS_4 . If there are in addition conserved higher spin currents in the CFT, the dual bulk theory would contain higher spin gauge fields. How parity even three point functions of higher spin currents arise from a bulk higher spin gauge theory was understood in [5, 6]. Here we will consider a few possible terms in the bulk Lagrangian that could give rise to the parity odd three point functions.

Consider first the parity odd structure of the three point function of the stress energy tensor, $\langle TTT \rangle$. In the bulk Lagrangian, the parity odd term with the least number of derivatives, that is not a total derivative, and does not vanish by the equation of motion, is expressed in terms of Weyl tensor as

$$(W_+)^3 - (W_-)^3 = W_{\alpha\beta}{}^{\gamma\delta} W_{\gamma\delta}{}^{\sigma\tau} W_{\sigma\tau}{}^{\alpha\beta} - W_{\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}\dot{\delta}} W_{\dot{\gamma}\dot{\delta}}{}^{\dot{\sigma}\dot{\tau}} W_{\dot{\sigma}\dot{\tau}}{}^{\dot{\alpha}\dot{\beta}} \quad (4.1)$$

where α, β, \dots are chiral spinor indices and $\dot{\alpha}, \dot{\beta}, \dots$ are anti-chiral spinor indices. This term is expected to contribute to the parity odd $\langle TTT \rangle$ structure in the boundary theory. Similarly, there is a parity violating graviton-vector field coupling

$$W_+ F_+^2 - W_- F_-^2 = W_{\alpha\beta\gamma\delta} F^{\alpha\beta} F^{\gamma\delta} - W_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} F^{\dot{\alpha}\dot{\beta}} F^{\dot{\gamma}\dot{\delta}} \quad (4.2)$$

that could give rise to the parity odd $\langle Tjj \rangle$ structure we found.

For higher spin gauge fields, one may consider bulk cubic terms involving fields of spins s_1, s_2 , and s_3 , of the form

$$W_+^{(s_1)}(y) * W_+^{(s_2)}(y) * W_+^{(s_3)}(y) \Big|_{y=0} - W_-^{(s_1)}(\bar{y}) * W_-^{(s_2)}(\bar{y}) * W_-^{(s_3)}(\bar{y}) \Big|_{\bar{y}=0}, \quad (4.3)$$

where

$$W_+(y) = \sum W_{\alpha_1 \dots \alpha_{2s}} y^{\alpha_1} \dots y^{\alpha_{2s}}, \quad W_-(\bar{y}) = \sum W_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}} \bar{y}^{\dot{\alpha}_1} \dots \bar{y}^{\dot{\alpha}_{2s}}. \quad (4.4)$$

Here $W_{\alpha_1 \dots \alpha_{2s}}$ are the self dual components of the generalized Weyl tensor of the spin s field, and similarly $W_{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}$ the anti self dual components. The $*$ product is defined as in [7]. Here it simply contracts y^α with y^β , giving $\epsilon^{\alpha\beta}$, and similarly on the \bar{y} 's. While we expect (4.3) to give a parity odd $\langle JJJ \rangle$ structure when the three spins s_1, s_2, s_3 obey triangular inequality, (4.3) vanishes when the three spins do not obey triangular inequality. This is precisely in agreement with the absence of parity odd, conformally invariant and conserved $\langle J_{s_1} J_{s_2} J_{s_3} \rangle$ structure, as discussed in earlier sections.¹⁰ A more detailed exploration of such parity violating bulk couplings of higher spin fields and in particular their connection to Vasiliev's higher spin gauge theory in AdS_4 will be given elsewhere.

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¹⁰Though, naively, it appears that one can write down parity odd cubic couplings of higher spin Weyl tensors violating triangular inequality, by acting with covariant derivatives. Nonetheless, our previous analysis indicates that such terms cannot contribute to the three point function as forbidden by conformal symmetry and higher spin gauge symmetry (current conservation in the boundary theory).

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A Calculation of $\langle j^a j^b j^c \rangle$ in vector notation

In this appendix, we obtain the allowed terms for the correlation function $\langle j_\mu^a(x_1) j_\nu^b(x_2) j_\omega^c(x_3) \rangle$, which appears to be the simplest correlation function admitting a new parity-violating structure, using the formalism of Osborn and Petkou [8]. This illustrates how the existence of parity-violating terms can be seen in their formalism, and provides a simple cross-check of our approach.

Following [8], the correlation function must take the form:

$$\langle j_\mu^a(x_1) j_\nu^b(x_2) j_\omega^c(x_3) \rangle = \frac{f^{abc}}{x_{12}^2 x_{23}^2 x_{13}^2} I_{\mu\alpha}(x_{13}) I_{\nu\beta}(x_{23}) t_{\alpha\beta\omega}(X_{12}) \quad (\text{A.1})$$

where $X_{12} = \frac{x_{13}}{x_{13}^2} - \frac{x_{23}}{x_{23}^2}$ and $I_{\mu\alpha}(X) = \delta_{\mu\alpha} - 2X_\mu X_\alpha X^{-2}$. These are related to the parity-invariant objects defined in the main text via:

$$Q_1 = 2\varepsilon_1^\mu X_{23\mu}, \quad (\text{A.2})$$

$$(P_3)^2 = -2\varepsilon_1^\mu \varepsilon_2^\nu x_{12}^{-2} I_{\mu\nu}(x_{12}) \quad (\text{A.3})$$

The quantity $t_{\alpha\beta\omega}$ must have scaling dimension zero and satisfy:

$$t_{\mu\nu\omega}(X) = -t_{\nu\mu\omega}(-X) \quad (\text{A.4})$$

$$I_{\mu\alpha} t_{\alpha\nu\omega}(X) = -t_{\omega\mu\nu}(X) \quad (\text{A.5})$$

$$\left(\partial_\mu - 2 \frac{X_\mu}{X^2} \right) t_{\mu\nu\omega}(X) = 0 \quad (\text{A.6})$$

The first and second conditions ensure that the correlation function has the required symmetries, the second condition also effectively ensures that the correlation function is invariant under inversions (up to a minus sign for parity-violating terms), and the third condition is due to current conservation. The signs in equations (A.4) and (A.5), which originate from equations (2.20) and (2.21) of [8], have been carefully arranged to allow both parity-preserving and parity-violating solutions.

The general solution for $t_{\alpha\beta\omega}$ given in [8] satisfying these conditions is:

$$t_{\mu\nu\omega}(X) = a_1 \frac{X_\mu X_\nu X_\omega}{X^3} + a_2 (X_\mu \delta_{\nu\omega} + X_\nu \delta_{\mu\omega} - X_\omega \delta_{\mu\nu}) X^{-1} \quad (\text{A.7})$$

Including terms involving $\epsilon_{\mu\nu\omega}$, we find that

$$t_{\mu\nu\omega}(X) = b (-X_\mu X^\lambda \epsilon_{\nu\omega\lambda} X^{-2} + X_\nu X^\lambda \epsilon_{\mu\omega\lambda} X^{-2} + X_\omega X^\lambda \epsilon_{\mu\nu\lambda} X^{-2}) \quad (\text{A.8})$$

is another solution to the above equations.

Translating into our notation, the correlation function is:

$$\begin{aligned} \langle j_\mu^a(x_1) j_\nu^b(x_2) j_\omega^c(x_3) \rangle \varepsilon_1^\mu \varepsilon_2^\nu \varepsilon_3^\omega = \\ \frac{f^{abc}}{|x_{12}| |x_{23}| |x_{31}|} \left(\frac{a_1}{8} Q_1 Q_2 Q_3 + \frac{a_2}{4} ((P_1)^2 Q_1 + (P_2)^2 Q_2 + (P_3)^2 Q_3 - Q_1 Q_2 Q_3) \right. \\ \left. + b(Q_1 S_1 + Q_2 S_2 + Q_3 S_3) \right) \end{aligned}$$

Indeed, using the nonlinear constraint (2.14), it is clear that the only allowed parity-preserving tensor structures must be linear combinations of $Q_1 Q_2 Q_3$ and $(P_1)^2 Q_1 + (P_2)^2 Q_2 + (P_3)^2 Q_3$. It is also easy to see that the only allowed parity-violating tensor structure is $Q_1 S_1 + Q_2 S_2 + Q_3 S_3$.

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